METRIC VERSION

CALCULUS

METRIC VERSION | 9E

Multivariable Calculus

James Stewart
Daniel Clegg
Saleem Watson

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REFERENCE page 1

ALGEBRA

Arithmetic Operations

$$a(b+c) = ab + ac \qquad \frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$

$$\frac{a+c}{b} = \frac{a}{b} + \frac{c}{b} \qquad \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$$

Exponents and Radicals

$$x^m x^n = x^{m+n} \qquad \frac{x^m}{x^n} = x^{m-n}$$

$$(x^m)^n = x^{mn} \qquad x^{-n} = \frac{1}{x^n}$$

$$(xy)^n = x^n y^n \qquad \left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$$

$$x^{1/n} = \sqrt[n]{x} \qquad x^{m/n} = \sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m$$

$$\sqrt[n]{xy} = \sqrt[n]{x}\,\sqrt[n]{y} \qquad \sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$$

Factoring Special Polynomials

$$x^2 - y^2 = (x+y)(x-y)$$

$$x^3 + y^3 = (x+y)(x^2 - xy + y^2)$$

$$x^3 - y^3 = (x-y)(x^2 + xy + y^2)$$

Binomial Theorem

$$(x+y)^2 = x^2 + 2xy + y^2 \qquad (x-y)^2 = x^2 - 2xy + y^2$$

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$$

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2}x^{n-2}y^2 + \cdots + \binom{n}{k}x^{n-k}y^k + \cdots + nxy^{n-1} + y^n$$

where $\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot k}$

Quadratic Formula

If $ax^2 + bx + c = 0$, then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Inequalities and Absolute Value

If $a < b$ and $b < c$, then $a < c$.

If $a < b$, then $a + c < b + c$.

If $a < b$ and $c > 0$, then $ca < cb$.

If $a < b$ and $c < 0$, then $ca > cb$.

If $a > 0$, then

$|x| = a$ means $x = a$ or $x = -a$

$|x| < a$ means $-a < x < a$

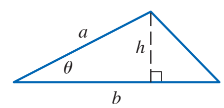
$|x| > a$ means $x > a$ or $x < -a$

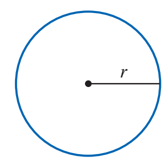
GEOMETRY

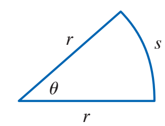
Geometric Formulas

Formulas for area A, circumference C, and volume V:

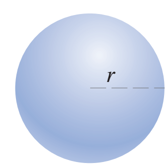
Triangle	Circle	Sector of Circle
$A = \frac{1}{2}bh$	$A = \pi r^2$	$A = \frac{1}{2}r^2\theta$
$= \frac{1}{2}ab\sin\theta$	$C = 2\pi r$	$s = r\theta$ (θ in radians)

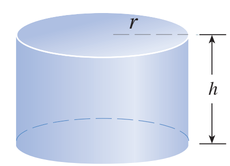


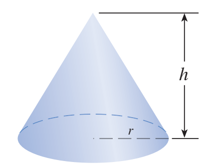




Sphere	Cylinder	Cone
$V = \frac{4}{3}\pi r^3$	$V = \pi r^2 h$	$V = \frac{1}{3}\pi r^2 h$
$A = 4\pi r^2$		$A = \pi r\sqrt{r^2 + h^2}$







Distance and Midpoint Formulas

Distance between $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Midpoint of $\overline{P_1P_2}$: $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Lines

Slope of line through $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Point-slope equation of line through $P_1(x_1, y_1)$ with slope m:

$$y - y_1 = m(x - x_1)$$

Slope-intercept equation of line with slope m and y-intercept b:

$$y = mx + b$$

Circles

Equation of the circle with center (h, k) and radius r:

$$(x-h)^2 + (y-k)^2 = r^2$$

REFERENCE page 2

TRIGONOMETRY

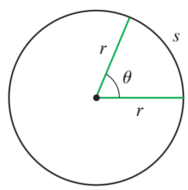
Angle Measurement

π radians $= 180°$

$1° = \dfrac{\pi}{180}$ rad $\qquad$ 1 rad $= \dfrac{180°}{\pi}$

$s = r\theta$

(θ in radians)

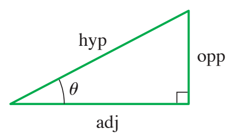


Right Angle Trigonometry

$\sin\theta = \dfrac{\text{opp}}{\text{hyp}}$ $\qquad$ $\csc\theta = \dfrac{\text{hyp}}{\text{opp}}$

$\cos\theta = \dfrac{\text{adj}}{\text{hyp}}$ $\qquad$ $\sec\theta = \dfrac{\text{hyp}}{\text{adj}}$

$\tan\theta = \dfrac{\text{opp}}{\text{adj}}$ $\qquad$ $\cot\theta = \dfrac{\text{adj}}{\text{opp}}$

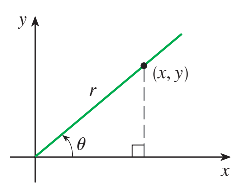


Trigonometric Functions

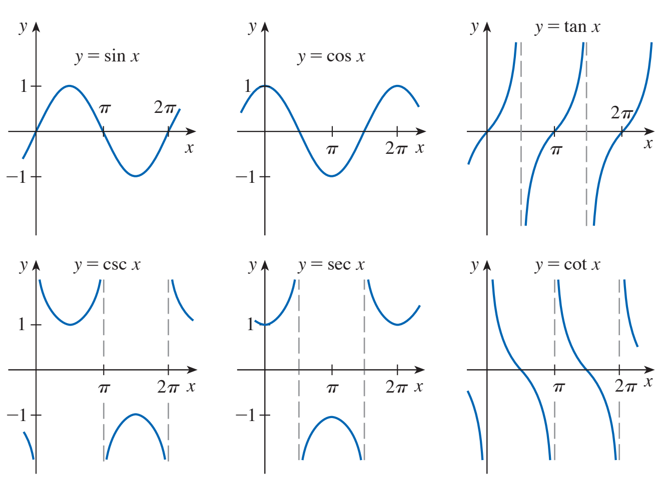
$\sin\theta = \dfrac{y}{r}$ $\qquad$ $\csc\theta = \dfrac{r}{y}$

$\cos\theta = \dfrac{x}{r}$ $\qquad$ $\sec\theta = \dfrac{r}{x}$

$\tan\theta = \dfrac{y}{x}$ $\qquad$ $\cot\theta = \dfrac{x}{y}$



Graphs of Trigonometric Functions



Trigonometric Functions of Important Angles

θ	radians	$\sin\theta$	$\cos\theta$	$\tan\theta$
$0°$	0	0	1	0
$30°$	$\pi/6$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$
$45°$	$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1
$60°$	$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
$90°$	$\pi/2$	1	0	—

Fundamental Identities

$\csc\theta = \dfrac{1}{\sin\theta}$ $\qquad$ $\sec\theta = \dfrac{1}{\cos\theta}$

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ $\qquad$ $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

$\cot\theta = \dfrac{1}{\tan\theta}$ $\qquad$ $\sin^2\theta + \cos^2\theta = 1$

$1 + \tan^2\theta = \sec^2\theta$ $\qquad$ $1 + \cot^2\theta = \csc^2\theta$

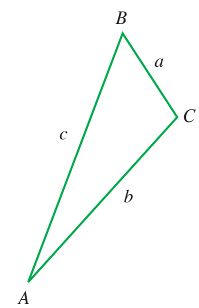
$\sin(-\theta) = -\sin\theta$ $\qquad$ $\cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta$ $\qquad$ $\sin\left(\dfrac{\pi}{2} - \theta\right) = \cos\theta$

$\cos\left(\dfrac{\pi}{2} - \theta\right) = \sin\theta$ $\qquad$ $\tan\left(\dfrac{\pi}{2} - \theta\right) = \cot\theta$

The Law of Sines

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



The Law of Cosines

$a^2 = b^2 + c^2 - 2bc\cos A$

$b^2 = a^2 + c^2 - 2ac\cos B$

$c^2 = a^2 + b^2 - 2ab\cos C$

Addition and Subtraction Formulas

$\sin(x + y) = \sin x\cos y + \cos x\sin y$

$\sin(x - y) = \sin x\cos y - \cos x\sin y$

$\cos(x + y) = \cos x\cos y - \sin x\sin y$

$\cos(x - y) = \cos x\cos y + \sin x\sin y$

$\tan(x + y) = \dfrac{\tan x + \tan y}{1 - \tan x\tan y}$

$\tan(x - y) = \dfrac{\tan x - \tan y}{1 + \tan x\tan y}$

Double-Angle Formulas

$\sin 2x = 2\sin x\cos x$

$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

$\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$

Half-Angle Formulas

$\sin^2 x = \dfrac{1 - \cos 2x}{2}$ $\qquad$ $\cos^2 x = \dfrac{1 + \cos 2x}{2}$

MULTIVARIABLE CALCULUS

NINTH EDITION

Metric Version

JAMES STEWART
McMASTER UNIVERSITY
AND
UNIVERSITY OF TORONTO

DANIEL CLEGG
PALOMAR COLLEGE

SALEEM WATSON
CALIFORNIA STATE UNIVERSITY, LONG BEACH

CENGAGE

Australia • Brazil • Mexico • Singapore • United Kingdom • United States

CENGAGE

***Multivariable Calculus*, Ninth Edition, Metric Version**

James Stewart, Daniel Clegg, Saleem Watson

Metric Version Prepared by Anthony Tan and Michael Verwer both at McMaster University

International Product Director, Global Editions: Timothy L. Anderson

Product Assistant: Andrew Reddish

Content Manager: Emma Collins

Production Service: Kathi Townes, TECHart

Compositor: Graphic World

Art Director: Angela Sheehan, Vernon Boes

IP Analyst: Ashley Maynard

IP Project Manager: Carly Belcher

Manager, Global IP Integration: Eleanor Rummer

Text Designer: Diane Beasley

Cover Designer: Nadine Ballard

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Appendixes A1

Preface

A great discovery solves a great problem but there is a grain of discovery in the solution of any problem. Your problem may be modest; but if it challenges your curiosity and brings into play your inventive faculties, and if you solve it by your own means, you may experience the tension and enjoy the triumph of discovery.

GEORGE POLYA

The art of teaching, Mark Van Doren said, is the art of assisting discovery. In this Ninth Edition, Metric Version, as in all of the preceding editions, we continue the tradition of writing a book that, we hope, assists students in discovering calculus—both for its practical power and its surprising beauty. We aim to convey to the student a sense of the utility of calculus as well as to promote development of technical ability. At the same time, we strive to give some appreciation for the intrinsic beauty of the subject. Newton undoubtedly experienced a sense of triumph when he made his great discoveries. We want students to share some of that excitement.

The emphasis is on understanding concepts. Nearly all calculus instructors agree that conceptual understanding should be the ultimate goal of calculus instruction; to implement this goal we present fundamental topics graphically, numerically, algebraically, and verbally, with an emphasis on the relationships between these different representations. Visualization, numerical and graphical experimentation, and verbal descriptions can greatly facilitate conceptual understanding. Moreover, conceptual understanding and technical skill can go hand in hand, each reinforcing the other.

We are keenly aware that good teaching comes in different forms and that there are different approaches to teaching and learning calculus, so the exposition and exercises are designed to accommodate different teaching and learning styles. The features (including projects, extended exercises, principles of problem solving, and historical insights) provide a variety of enhancements to a central core of fundamental concepts and skills. Our aim is to provide instructors and their students with the tools they need to chart their own paths to discovering calculus.

Alternate Versions

The Stewart *Calculus* series includes several other calculus textbooks that might be preferable for some instructors. Most of them also come in single variable and multivariable versions.

- *Calculus*, Ninth Edition, Metric Version, includes the material in this book as well as the single-variable calculus chapters. The exponential, logarithmic, and inverse trigonometric functions are covered after the chapter on integration.

- *Calculus: Early Transcendentals*, Ninth Edition, Metric Version, includes the material in this book in addition to single-variable calculus. The exponential, logarithmic, and inverse trigonometric functions are covered early, before the chapter on integration.

- *Essential Calculus*, Second Edition, is a much briefer book (840 pages), though it contains almost all of the topics in *Calculus*, Ninth Edition. The relative brevity is achieved through briefer exposition of some topics and putting some features on the website.
- *Essential Calculus: Early Transcendentals*, Second Edition, resembles *Essential Calculus*, but the exponential, logarithmic, and inverse trigonometric functions are covered in Chapter 3.
- *Calculus: Concepts and Contexts*, Fourth Edition, emphasizes conceptual understanding even more strongly than this book. The coverage of topics is not encyclopedic and the material on transcendental functions and on parametric equations is woven throughout the book instead of being treated in separate chapters.
- *Brief Applied Calculus* is intended for students in business, the social sciences, and the life sciences.
- *Biocalculus: Calculus for the Life Sciences* is intended to show students in the life sciences how calculus relates to biology.
- *Biocalculus: Calculus, Probability, and Statistics for the Life Sciences* contains all the content of *Biocalculus: Calculus for the Life Sciences* as well as three additional chapters covering probability and statistics.

What's New in the Ninth Edition, Metric Version?

The overall structure of the text remains largely the same, but we have made many improvements that are intended to make the Ninth Edition, Metric Version even more usable as a teaching tool for instructors and as a learning tool for students. The changes are a result of conversations with our colleagues and students, suggestions from users and reviewers, insights gained from our own experiences teaching from the book, and from the copious notes that James Stewart entrusted to us about changes that he wanted us to consider for the new edition. In all the changes, both small and large, we have retained the features and tone that have contributed to the success of this book.

- More than 20% of the exercises are new:

  Basic exercises have been added, where appropriate, near the beginning of exercise sets. These exercises are intended to build student confidence and reinforce understanding of the fundamental concepts of a section. (See, for instance, Exercises 11.4.3–6.)

  Some new exercises include graphs intended to encourage students to understand how a graph facilitates the solution of a problem; these exercises complement subsequent exercises in which students need to supply their own graph. (See Exercises 10.4.43–46 as well as 53–54, 15.5.1–2, 15.6.9–12, 16.7.15 and 24, 16.8.9 and 13.)

  Some exercises have been structured in two stages, where part (a) asks for the setup and part (b) is the evaluation. This allows students to check their answer to part (a) before completing the problem. (See Exercises 15.2.7–10.)

  Some challenging and extended exercises have been added toward the end of selected exercise sets (such as Exercises 11.2.79–81 and 11.9.47).

Titles have been added to selected exercises when the exercise extends a concept discussed in the section. (See, for example, Exercises 10.1.55–57 and 15.2.80–81.)

Some of our favorite new exercises are 10.5.69, 15.1.38, and 15.4.3–4. In addition, Problem 4 in the Problems Plus following Chapter 15 is interesting and challenging.

- New examples have been added, and additional steps have been added to the solutions of some existing examples. (See, for instance, Example 10.1.5, Examples 14.8.1 and 14.8.4, and Example 16.3.4.)
- Several sections have been restructured and new subheads added to focus the organization around key concepts. (Good illustrations of this are Sections 11.1, 11.2, and 14.2.)
- Many new graphs and illustrations have been added, and existing ones updated, to provide additional graphical insights into key concepts.
- A few new topics have been added and others expanded (within a section or in extended exercises) that were requested by reviewers. (See, for example, the subsection on torsion in Section 13.3.)
- New projects have been added and some existing projects have been updated. (For instance, see the Discovery Project following Section 12.2, *The Shape of a Hanging Chain*.)
- Alternating series and absolute convergence are now covered in one section (11.5).
- The chapter on Second-Order Differential Equations, as well as the associated appendix section on complex numbers, has been moved to the website.

Features

Each feature is designed to complement different teaching and learning practices. Throughout the text there are historical insights, extended exercises, projects, problem-solving principles, and many opportunities to experiment with concepts by using technology. We are mindful that there is rarely enough time in a semester to utilize all of these features, but their availability in the book gives the instructor the option to assign some and perhaps simply draw attention to others in order to emphasize the rich ideas of calculus and its crucial importance in the real world.

Conceptual Exercises

The most important way to foster conceptual understanding is through the problems that the instructor assigns. To that end we have included various types of problems. Some exercise sets begin with requests to explain the meanings of the basic concepts of the section (see, for instance, the first few exercises in Sections 11.2, 14.2, and 14.3) and most exercise sets contain exercises designed to reinforce basic understanding (such as Exercises 11.4.3–6). Other exercises test conceptual understanding through graphs or tables (see Exercises 10.1.30–33, 13.2.1–2, 13.3.37–43, 14.1.41–44, 14.3.2, 14.3.4–6, 14.6.1–2, 14.7.3–4, 15.1.6–8, 16.1.13–22, 16.2.19–20, and 16.3.1–2).

Many exercises provide a graph to aid in visualization (see for instance Exercises 10.4.43–46, 15.5.1–2, 15.6.9–12, and 16.7.24). In addition, all the review sections begin with a Concept Check and a True-False Quiz.

We particularly value problems that combine and compare different approaches (see Exercises 14.2.3–4, 14.7.3–4, 14.8.2, 15.4.3–4, and 16.3.13).

Graded Exercise Sets

Each exercise set is carefully graded, progressing from basic conceptual exercises, to skill-development and graphical exercises, and then to more challenging exercises that often extend the concepts of the section, draw on concepts from previous sections, or involve applications or proofs.

Real-World Data

Real-world data provide a tangible way to introduce, motivate, or illustrate the concepts of calculus. As a result, many of the examples and exercises deal with functions defined by such numerical data or graphs. These real-world data have been obtained by contacting companies and government agencies as well as researching on the Internet and in libraries. See, for instance, Example 3 in Section 14.4 (the heat index), Figure 1 in Section 14.6 (temperature contour map), Example 9 in Section 15.1 (snowfall in Colorado), and Figure 1 in Section 16.1 (velocity vector fields of wind in San Francisco Bay).

Projects

One way of involving students and making them active learners is to have them work (perhaps in groups) on extended projects that give a feeling of substantial accomplishment when completed. There are three kinds of projects in the text.

Applied Projects involve applications that are designed to appeal to the imagination of students. The project after Section 14.8 uses Lagrange multipliers to determine the masses of the three stages of a rocket so as to minimize the total mass while enabling the rocket to reach a desired velocity.

Discovery Projects anticipate results to be discussed later or encourage discovery through pattern recognition. Several discovery projects explore aspects of geometry: tetrahedra (after Section 12.4), hyperspheres (after Section 15.6), and intersections of three cylinders (after Section 15.7). Additionally, the project following Section 12.2 uses the geometric definition of the derivative to find a formula for the shape of a hanging chain. Some projects make substantial use of technology; the one following Section 10.2 shows how to use Bézier curves to design shapes that represent letters for a laser printer.

The *Writing Project* following Section 11.10 asks students to compare present-day methods with those of the founders of calculus. Suggested references are supplied.

More projects can be found in the *Instructor's Guide*. There are also extended exercises that can serve as smaller projects. (See Exercise 13.3.75 on the evolute of a curve, Exercise 14.7.61 on the method of least squares, or Exercise 16.3.42 on inverse square fields.)

Technology

When using technology, it is particularly important to clearly understand the concepts that underlie the images on the screen or the results of a calculation. When properly used, graphing calculators and computers are powerful tools for discovering and understanding those concepts. This textbook can be used either with or without technology—we use two special symbols to indicate clearly when a particular type of assistance from technology is required. The icon indicates an exercise that definitely requires the use of graphing software or a graphing calculator to aid in sketching a graph. (That is not to say that the technology can't be used on the other exercises as well.) The symbol T means that the assistance of software or a graphing calculator is needed beyond just graphing to complete the exercise. Freely available websites such as WolframAlpha.com or Symbolab.com are often suitable. In cases where the full resources of a computer algebra system, such as Maple or Mathematica, are needed, we state this in the exercise. Of course, technology doesn't make pencil and paper obsolete. Hand calculation and sketches are often preferable to technology for illustrating and reinforcing some concepts. Both instructors and students need to develop the ability to decide where using technology is appropriate and where more insight is gained by working out an exercise by hand.

CENGAGE | WEBASSIGN

WebAssign: webassign.net

This Ninth Edition is available with WebAssign, a fully customizable online solution for STEM disciplines from Cengage. WebAssign includes homework, an interactive mobile eBook, videos, tutorials and Explore It interactive learning modules. Instructors can decide what type of help students can access, and when, while working on assignments. The patented grading engine provides unparalleled answer evaluation, giving students instant feedback, and insightful analytics highlight exactly where students are struggling. For more information, visit cengage.com/WebAssign.

Stewart Website

Visit StewartCalculus.com for these additional materials:

- Homework Hints
- Solutions to the Concept Checks (from the review section of each chapter)
- Algebra and Analytic Geometry Review
- Lies My Calculator and Computer Told Me
- History of Mathematics, with links to recommended historical websites
- Additional Topics (complete with exercise sets): Fourier Series, Rotation of Axes, Formulas for the Remainder Theorem in Taylor Series
- Additional chapter on second-order differential equations, including the method of series solutions, and an appendix section reviewing complex numbers and complex exponential functions
- Instructor Area that includes archived problems (drill exercises that appeared in previous editions, together with their solutions)
- Challenge Problems (some from the Problems Plus sections from prior editions)
- Links, for particular topics, to outside Web resources

Content

10 Parametric Equations and Polar Coordinates

This chapter introduces parametric and polar curves and applies the methods of calculus to them. Parametric curves are well suited to projects that require graphing with technology; the two presented here involve families of curves and Bézier curves. A brief treatment of conic sections in polar coordinates prepares the way for Kepler's Laws in Chapter 13.

11 Sequences, Series, and Power Series

The convergence tests have intuitive justifications (see Section 11.3) as well as formal proofs. Numerical estimates of sums of series are based on which test was used to prove convergence. The emphasis is on Taylor series and polynomials and their applications to physics.

12 Vectors and the Geometry of Space The material on three-dimensional analytic geometry and vectors is covered in this and the next chapter. Here we deal with vectors, the dot and cross products, lines, planes, and surfaces.

13 Vector Functions This chapter covers vector-valued functions, their derivatives and integrals, the length and curvature of space curves, and velocity and acceleration along space curves, culminating in Kepler's laws.

14 Partial Derivatives Functions of two or more variables are studied from verbal, numerical, visual, and algebraic points of view. In particular, partial derivatives are introduced by looking at a specific column in a table of values of the heat index (perceived air temperature) as a function of the actual temperature and the relative humidity.

15 Multiple Integrals Contour maps and the Midpoint Rule are used to estimate the average snowfall and average temperature in given regions. Double and triple integrals are used to compute volumes, surface areas, and (in projects) volumes of hyperspheres and volumes of intersections of three cylinders. Cylindrical and spherical coordinates are introduced in the context of evaluating triple integrals. Several applications are considered, including computing mass, charge, and probabilities.

16 Vector Calculus Vector fields are introduced through pictures of velocity fields showing San Francisco Bay wind patterns. The similarities among the Fundamental Theorem for line integrals, Green's Theorem, Stokes' Theorem, and the Divergence Theorem are emphasized.

17 Second-Order Differential Equations Since first-order differential equations are covered in Chapter 9, this online chapter deals with second-order linear differential equations, their application to vibrating springs and electric circuits, and series solutions.

Ancillaries

Multivariable Calculus, Ninth Edition, Metric Version is supported by a complete set of ancillaries. Each piece has been designed to enhance student understanding and to facilitate creative instruction.

Ancillaries for Instructors

Instructor's Guide

by Douglas Shaw

Each section of the text is discussed from several viewpoints. Available online at the Instructor's Companion Site, the Instructor's Guide *contains suggested time to allot, points to stress, text discussion topics, core materials for lecture, workshop/discussion suggestions, group work exercises in a form suitable for handout, and suggested homework assignments.*

Complete Solutions Manual

***Multivariable Calculus*, Ninth Edition, Metric Version**
Chapters 10–16
By Joshua Babbin and Gina Sanders with metrication by Anthony Tan and Mike Verwer both from McMaster University

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Acknowledgments

One of the main factors aiding in the preparation of this edition is the cogent advice from a large number of reviewers, all of whom have extensive experience teaching calculus. We greatly appreciate their suggestions and the time they spent to understand the approach taken in this book. We have learned something from each of them.

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This textbook has benefited greatly over the past three decades from the advice and guidance of some of the best mathematics editors: Ron Munro, Harry Campbell, Craig Barth, Jeremy Hayhurst, Gary Ostedt, Bob Pirtle, Richard Stratton, Liz Covello, Neha Taleja, and now Gary Whalen. They have all contributed significantly to the success of this book. Prominently, Gary Whalen's broad knowledge of current issues in the teaching of mathematics and his continual research into creating better ways of using technology as a teaching and learning tool were invaluable resources in the creation of this edition.

JAMES STEWART
DANIEL CLEGG
SALEEM WATSON

Technology in the Ninth Edition

Graphing and computing devices are valuable tools for learning and exploring calculus, and some have become well established in calculus instruction. Graphing calculators are useful for drawing graphs and performing some numerical calculations, like approximating solutions to equations or numerically evaluating derivatives or definite integrals. Mathematical software packages called computer algebra systems (CAS, for short) are more powerful tools. Despite the name, algebra represents only a small subset of the capabilities of a CAS. In particular, a CAS can do mathematics symbolically rather than just numerically. It can find exact solutions to equations and exact formulas for derivatives and integrals.

We now have access to a wider variety of tools of varying capabilities than ever before. These include Web-based resources (some of which are free of charge) and apps for smartphones and tablets. Many of these resources include at least some CAS functionality, so some exercises that may have typically required a CAS can now be completed using these alternate tools.

In this edition, rather than refer to a specific type of device (a graphing calculator, for instance) or software package (such as a CAS), we indicate the type of capability that is needed to work an exercise.

Graphing Icon

The appearance of this icon beside an exercise indicates that you are expected to use a machine or software to help you draw the graph. In many cases, a graphing calculator will suffice. Websites such as Desmos.com provide similar capability. If the graph is in 3D (see Chapters 12–16), WolframAlpha.com is a good resource. There are also many graphing software applications for computers, smartphones, and tablets. If an exercise asks for a graph but no graphing icon is shown, then you are expected to draw the graph by hand.

T Technology Icon

This icon is used to indicate that software or a device with abilities beyond just graphing is needed to complete the exercise. Many graphing calculators and software resources can provide numerical approximations when needed. For working with mathematics symbolically, websites like WolframAlpha.com or Symbolab.com are helpful, as are more advanced graphing calculators such as the Texas Instrument TI-89 or TI-Nspire CAS. If the full power of a CAS is needed, this will be stated in the exercise, and access to software packages such as Mathematica, Maple, MATLAB, or SageMath may be required. If an exercise does not include a technology icon, then you are expected to evaluate limits, derivatives, and integrals, or solve equations by hand, arriving at exact answers. No technology is needed for these exercises beyond perhaps a basic scientific calculator.

To the Student

Reading a calculus textbook is different from reading a story or a news article. Don't be discouraged if you have to read a passage more than once in order to understand it. You should have pencil and paper and calculator at hand to sketch a diagram or make a calculation.

Some students start by trying their homework problems and read the text only if they get stuck on an exercise. We suggest that a far better plan is to read and understand a section of the text before attempting the exercises. In particular, you should look at the definitions to see the exact meanings of the terms. And before you read each example, we suggest that you cover up the solution and try solving the problem yourself.

Part of the aim of this course is to train you to think logically. Learn to write the solutions of the exercises in a connected, step-by-step fashion with explanatory sentences—not just a string of disconnected equations or formulas.

The answers to the odd-numbered exercises appear at the back of the book, in Appendix H. Some exercises ask for a verbal explanation or interpretation or description. In such cases there is no single correct way of expressing the answer, so don't worry that you haven't found the definitive answer. In addition, there are often several different forms in which to express a numerical or algebraic answer, so if your answer differs from the given one, don't immediately assume you're wrong. For example, if the answer given in the back of the book is $\sqrt{2} - 1$ and you obtain $1/(1 + \sqrt{2})$, then you're correct and rationalizing the denominator will show that the answers are equivalent.

The icon indicates an exercise that definitely requires the use of either a graphing calculator or a computer with graphing software to help you sketch the graph. But that doesn't mean that graphing devices can't be used to check your work on the other exercises as well. The symbol **T** indicates that technological assistance beyond just graphing is needed to complete the exercise. (See Technology in the Ninth Edition for more details.)

You will also encounter the symbol ⊘, which warns you against committing an error. This symbol is placed in the margin in situations where many students tend to make the same mistake.

Homework Hints are available for many exercises. These hints can be found on StewartCalculus.com as well as in WebAssign. The homework hints ask you questions that allow you to make progress toward a solution without actually giving you the answer. If a particular hint doesn't enable you to solve the problem, you can click to reveal the next hint.

We recommend that you keep this book for reference purposes after you finish the course. Because you will likely forget some of the specific details of calculus, the book will serve as a useful reminder when you need to use calculus in subsequent courses. And, because this book contains more material than can be covered in any one course, it can also serve as a valuable resource for a working scientist or engineer.

Calculus is an exciting subject, justly considered to be one of the greatest achievements of the human intellect. We hope you will discover that it is not only useful but also intrinsically beautiful.

The photo shows comet Hale-Bopp as it passed the earth in 1997, due to return in 4380. One of the brightest comets of the past century, Hale-Bopp could be observed in the night sky by the naked eye for about 18 months. It was named after its discoverers Alan Hale and Thomas Bopp, who first obse ved it by telescope in 1995 (Hale in New Mexico and Bopp in Arizona). In Section 10.6 you will see how polar coordinates provide a convenient equation for the elliptical path of the comet's orbit.

Jeff Schneiderman / Moment Open / Getty Images

10 Parametric Equations and Polar Coordinates

SO FAR WE HAVE DESCRIBED plane curves by giving y as a function of x [$y = f(x)$] or x as a function of y [$x = g(y)$] or by giving a relation between x and y that defines y implicitly as a function of x [$f(x, y) = 0$]. In this chapter we discuss two new methods for describing curves.

Some curves, such as the cycloid, are best handled when both x and y are given in terms of a third variable t called a parameter [$x = f(t)$, $y = g(t)$]. Other curves, such as the cardioid, have their most convenient description when we use a new coordinate system, called the polar coordinate system.

10.1 Curves Defined by Parametric Equations

Imagine that a particle moves along the curve C shown in Figure 1. It is impossible to describe C by an equation of the form $y = f(x)$ because C fails the Vertical Line Test. But the x- and y-coordinates of the particle are functions of time t and so we can write $x = f(t)$ and $y = g(t)$. Such a pair of equations is often a convenient way of describing a curve.

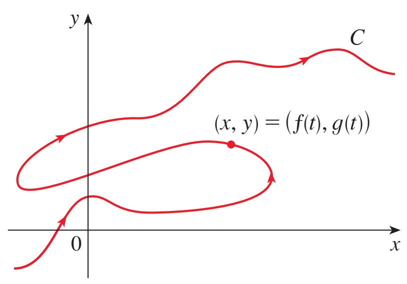


FIGURE 1

Parametric Equations

Suppose that x and y are both given as functions of a third variable t, called a **parameter**, by the equations

$$x = f(t) \qquad y = g(t)$$

which are called **parametric equations**. Each value of t determines a point (x, y), which we can plot in a coordinate plane. As t varies, the point $(x, y) = (f(t), g(t))$ varies and traces out a curve called a **parametric curve**. The parameter t does not necessarily represent time and, in fact, we could use a letter other than t for the parameter. But in many applications of parametric curves, t does denote time and in this case we can interpret $(x, y) = (f(t), g(t))$ as the position of a moving object at time t.

EXAMPLE 1 Sketch and identify the curve defined by the parametric equations

$$x = t^2 - 2t \qquad y = t + 1$$

SOLUTION Each value of t gives a point on the curve, as shown in the table. For instance, if $t = 1$, then $x = -1$, $y = 2$ and so the corresponding point is $(-1, 2)$. In Figure 2 we plot the points (x, y) determined by several values of the parameter and we join them to produce a curve.

t	x	y
-2	8	-1
-1	3	0
0	0	1
1	-1	2
2	0	3
3	3	4
4	8	5

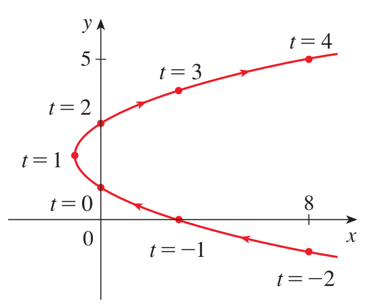


FIGURE 2

A particle whose position at time t is given by the parametric equations moves along the curve in the direction of the arrows as t increases. Notice that the consecutive points marked on the curve appear at equal time intervals but not at equal distances. That is because the particle slows down and then speeds up as t increases.

It appears from Figure 2 that the curve traced out by the particle may be a parabola. In fact, from the second equation we obtain $t = y - 1$ and substitution into the first equation gives

$$x = t^2 - 2t = (y - 1)^2 - 2(y - 1) = y^2 - 4y + 3$$

Since the equation $x = y^2 - 4y + 3$ is satisfied for all pairs of x- and y-values generated by the parametric equations, every point (x, y) on the parametric curve must lie on the parabola $x = y^2 - 4y + 3$ and so the parametric curve coincides with at least part of this parabola. Because t can be chosen to make y any real number, we know that the parametric curve is the entire parabola. ■

It is not always possible to eliminate the parameter from parametric equations. There are many parametric curves that don't have an equivalent representation as an equation in x and y.

In Example 1 we found a Cartesian equation in x and y whose graph coincided with the curve represented by parametric equations. This process is called **eliminating the parameter**; it can be helpful in identifying the shape of the parametric curve, but we lose some information in the process. The equation in x and y describes the curve the particle travels along, whereas the parametric equations have additional advantages—they tell us *where* the particle is at any given *time* and indicate the *direction* of motion. If you think of the graph of an equation in x and y as a road, then the parametric equations could track the motion of a car traveling along the road.

No restriction was placed on the parameter t in Example 1, so we assumed that t could be any real number (including negative numbers). But sometimes we restrict t to lie in a particular interval. For instance, the parametric curve

$$x = t^2 - 2t \qquad y = t + 1 \qquad 0 \leqslant t \leqslant 4$$

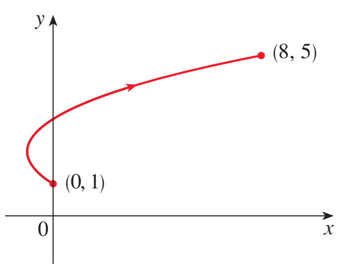


FIGURE 3

shown in Figure 3 is the part of the parabola in Example 1 that starts at the point (0, 1) and ends at the point (8, 5). The arrowhead indicates the direction in which the curve is traced as t increases from 0 to 4.

In general, the curve with parametric equations

$$x = f(t) \qquad y = g(t) \qquad a \leqslant t \leqslant b$$

has **initial point** $(f(a), g(a))$ and **terminal point** $(f(b), g(b))$.

EXAMPLE 2 What curve is represented by the following parametric equations?

$$x = \cos t \qquad y = \sin t \qquad 0 \leqslant t \leqslant 2\pi$$

SOLUTION If we plot points, it appears that the curve is a circle. We can confirm this by eliminating the parameter t. Observe that

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$$

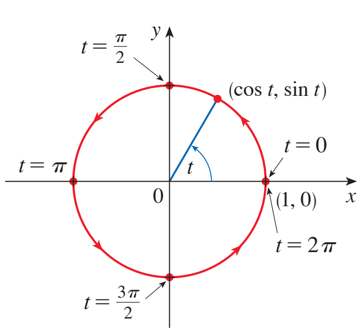


FIGURE 4

Because $x^2 + y^2 = 1$ is satisfied for all pairs of x- and y-values generated by the parametric equations, the point (x, y) moves along the unit circle $x^2 + y^2 = 1$. Notice that in this example the parameter t can be interpreted as the angle (in radians) shown in Figure 4. As t increases from 0 to 2π, the point $(x, y) = (\cos t, \sin t)$ moves once around the circle in the counterclockwise direction starting from the point (1, 0). ■

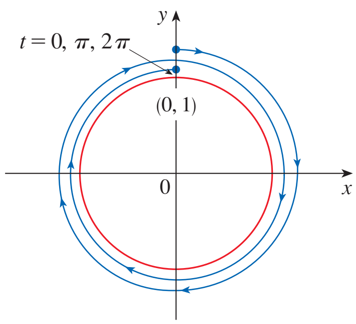


FIGURE 5

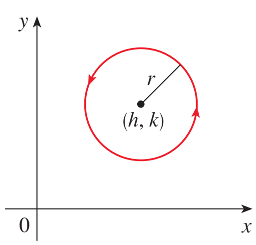


FIGURE 6
$x = h + r\cos t$, $y = k + r\sin t$

EXAMPLE 3 What curve is represented by the given parametric equations?

$$x = \sin 2t \qquad y = \cos 2t \qquad 0 \leqslant t \leqslant 2\pi$$

SOLUTION Again we have

$$x^2 + y^2 = \sin^2(2t) + \cos^2(2t) = 1$$

so the parametric equations again represent the unit circle $x^2 + y^2 = 1$. But as t increases from 0 to 2π, the point $(x, y) = (\sin 2t, \cos 2t)$ starts at $(0, 1)$ and moves *twice* around the circle in the clockwise direction as indicated in Figure 5. ■

EXAMPLE 4 Find parametric equations for the circle with center (h, k) and radius r.

SOLUTION One way is to take the parametric equations of the unit circle in Example 2 and multiply the expressions for x and y by r, giving $x = r\cos t$, $y = r\sin t$. You can verify that these equations represent a circle with radius r and center the origin, traced counterclockwise. We now shift h units in the x-direction and k units in the y-direction and obtain parametric equations of the circle (Figure 6) with center (h, k) and radius r:

$$x = h + r\cos t \qquad y = k + r\sin t \qquad 0 \leqslant t \leqslant 2\pi$$ ■

NOTE Examples 2 and 3 show that different parametric equations can represent the same curve. Thus we distinguish between a *curve*, which is a set of points, and a *parametric curve*, in which the points are traced out in a particular way.

In the next example we use parametric equations to describe the motions of four different particles traveling along the same curve but in different ways.

EXAMPLE 5 Each of the following sets of parametric equations gives the position of a moving particle at time t.

(a) $x = t^3, \quad y = t$ (b) $x = -t^3, \quad y = -t$

(c) $x = t^{3/2}, \quad y = \sqrt{t}$ (d) $x = e^{-3t}, \quad y = e^{-t}$

In each case, eliminating the parameter gives $x = y^3$, so each particle moves along the cubic curve $x = y^3$; however, the particles move in different ways, as illustrated in Figure 7.

(a) The particle moves from left to right as t increases.

(b) The particle moves from right to left as t increases.

(c) The equations are defined only for $t \geqslant 0$. The particle starts at the origin (where $t = 0$) and moves to the right as t increases.

(d) Here $x > 0$ and $y > 0$ for all t. The particle moves from right to left and approaches the point (1,1) as t increases (through negative values) toward 0. As t further increases, the particle approaches, but does not reach, the origin.

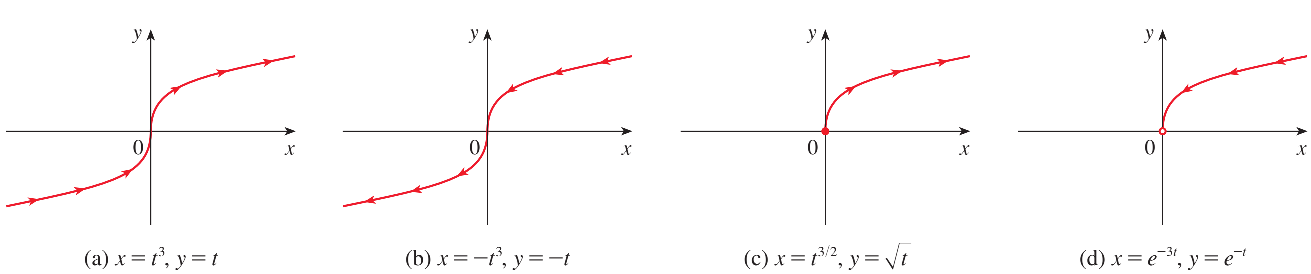


FIGURE 7 ■

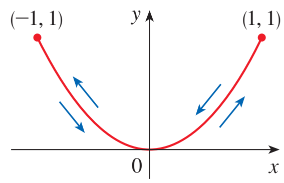
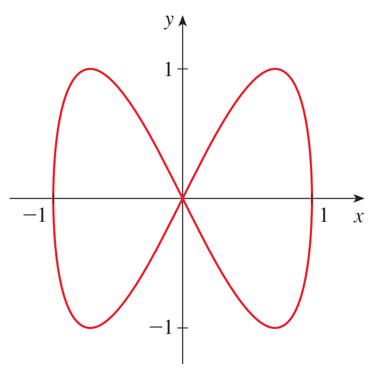


FIGURE 8

EXAMPLE 6 Sketch the curve with parametric equations $x = \sin t$, $y = \sin^2 t$.

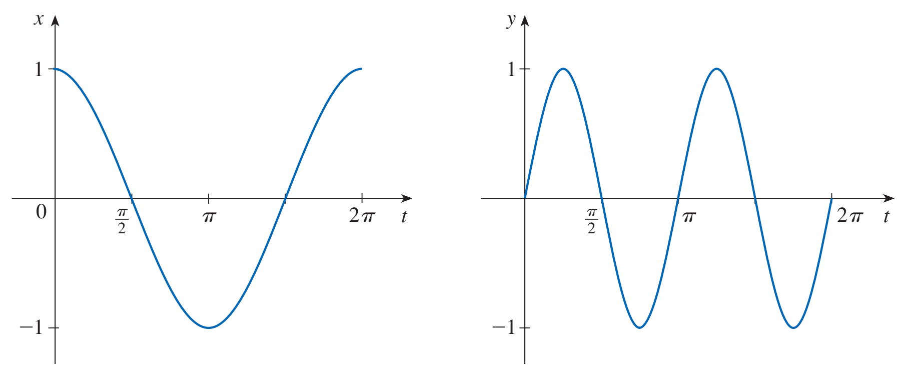
SOLUTION Observe that $y = (\sin t)^2 = x^2$ and so the point (x, y) moves on the parabola $y = x^2$. But note also that, since $-1 \leqslant \sin t \leqslant 1$, we have $-1 \leqslant x \leqslant 1$, so the parametric equations represent only the part of the parabola for which $-1 \leqslant x \leqslant 1$. Since $\sin t$ is periodic, the point $(x, y) = (\sin t, \sin^2 t)$ moves back and forth infinitely often along the parabola from $(-1, 1)$ to $(1, 1)$. (See Figure 8.) ■

EXAMPLE 7 The curve represented by the parametric equations $x = \cos t$, $y = \sin 2t$ is shown in Figure 9. It is an example of a *Lissajous figure* (see Exercise 63). It is possible to eliminate the parameter, but the resulting equation $(y^2 = 4x^2 - 4x^4)$ isn't very helpful. Another way to visualize the curve is to first draw graphs of x and y individually as functions of t, as shown in Figure 10.



$x = \cos t$, $y = \sin 2t$

FIGURE 9



$x = \cos t$ $\qquad$ $y = \sin 2t$

FIGURE 10

We see that as t increases from 0 to $\pi/2$, x decreases from 1 to 0 while y starts at 0, increases to 1, and then returns to 0. Together these descriptions produce the portion of the parametric curve that we see in the first quadrant. If we proceed similarly, we get the complete curve. (See Exercises 31–33 for practice with this technique.) ■

■ Graphing Parametric Curves with Technology

Most graphing software applications and graphing calculators can graph curves defined by parametric equations. In fact, it's instructive to watch a parametric curve being drawn by a graphing calculator because the points are plotted in order as the corresponding parameter values increase.

The next example shows that parametric equations can be used to produce the graph of a Cartesian equation where x is expressed as a function of y. (Some calculators, for instance, require y to be expressed as a function of x.)

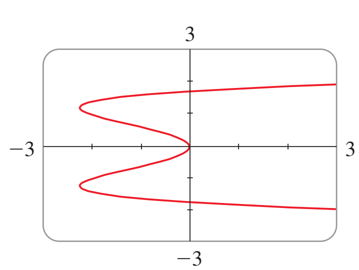


FIGURE 11

EXAMPLE 8 Use a calculator or computer to graph the curve $x = y^4 - 3y^2$.

SOLUTION If we let the parameter be $t = y$, then we have the equations

$$x = t^4 - 3t^2 \qquad y = t$$

Using these parametric equations to graph the curve, we obtain Figure 11. It would be possible to solve the given equation $(x = y^4 - 3y^2)$ for y as four functions of x and graph them individually, but the parametric equations provide a much easier method. ■

In general, to graph an equation of the form $x = g(y)$, we can use the parametric equations

$$x = g(t) \qquad y = t$$

In the same spirit, notice that curves with equations $y = f(x)$ (the ones we are most familiar with—graphs of functions) can also be regarded as curves with parametric equations

$$x = t \qquad y = f(t)$$

Graphing software is particularly useful for sketching complicated parametric curves. For instance, the curves shown in Figures 12, 13, and 14 would be virtually impossible to produce by hand.

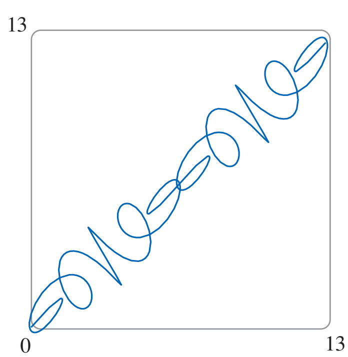


FIGURE 12
$x = t + \sin 5t$
$y = t + \sin 6t$

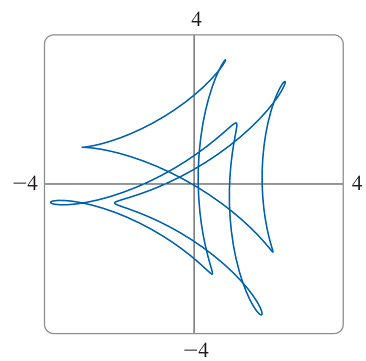


FIGURE 13
$x = \cos t + \cos 6t + 2 \sin 3t$
$y = \sin t + \sin 6t + 2 \cos 3t$

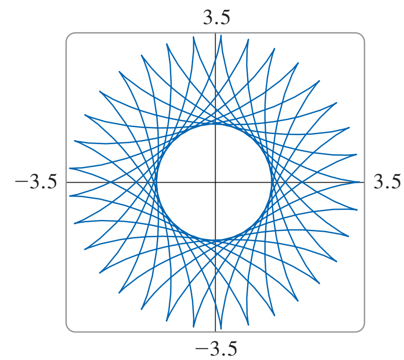


FIGURE 14
$x = 2.3 \cos 10t + \cos 23t$
$y = 2.3 \sin 10t - \sin 23t$

One of the most important uses of parametric curves is in computer-aided design (CAD). In the Discovery Project after Section 10.2 we will investigate special parametric curves, called **Bézier curves**, that are used extensively in manufacturing, especially in the automotive industry. These curves are also employed in specifying the shapes of letters and other symbols in PDF documents and laser printers.

The Cycloid

EXAMPLE 9 The curve traced out by a point P on the circumference of a circle as the circle rolls along a straight line is called a **cycloid**. (Think of the path traced out by a pebble stuck in a car tire; see Figure 15.) If the circle has radius r and rolls along the x-axis and if one position of P is the origin, find parametric equations for the cycloid.

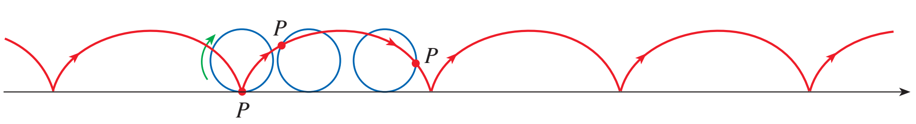


FIGURE 15

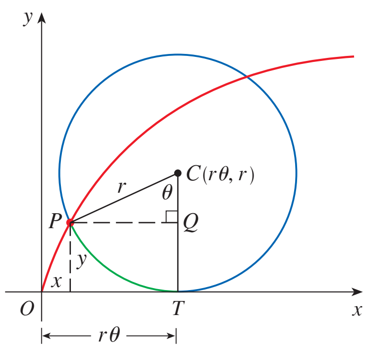


FIGURE 16

SOLUTION We choose as parameter the angle of rotation θ of the circle ($\theta = 0$ when P is at the origin). Suppose the circle has rotated through θ radians. Because the circle has been in contact with the line, we see from Figure 16 that the distance it has rolled from the origin is

$$|OT| = \text{arc } PT = r\theta$$

Therefore the center of the circle is $C(r\theta, r)$. Let the coordinates of P be (x, y). Then from Figure 16 we see that

$$x = |OT| - |PQ| = r\theta - r\sin\theta = r(\theta - \sin\theta)$$

$$y = |TC| - |QC| = r - r\cos\theta = r(1 - \cos\theta)$$

Therefore parametric equations of the cycloid are

$$\boxed{1} \qquad x = r(\theta - \sin\theta) \qquad y = r(1 - \cos\theta) \qquad \theta \in \mathbb{R}$$

One arch of the cycloid comes from one rotation of the circle and so is described by $0 \leqslant \theta \leqslant 2\pi$. Although Equations 1 were derived from Figure 16, which illustrates the case where $0 < \theta < \pi/2$, it can be seen that these equations are still valid for other values of θ (see Exercise 48).

Although it is possible to eliminate the parameter θ from Equations 1, the resulting Cartesian equation in x and y is very complicated [$x = r\cos^{-1}(1 - y/r) - \sqrt{2ry - y^2}$ gives just half of one arch] and not as convenient to work with as the parametric equations. ■

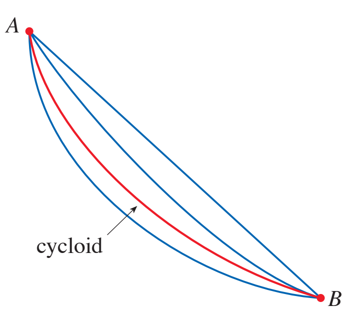


FIGURE 17

One of the first people to study the cycloid was Galileo; he proposed that bridges be built in the shape of cycloids and tried to find the area under one arch of a cycloid. Later this curve arose in connection with the **brachistochrone problem**: Find the curve along which a particle will slide in the shortest time (under the influence of gravity) from a point A to a lower point B not directly beneath A. The Swiss mathematician John Bernoulli, who posed this problem in 1696, showed that among all possible curves that join A to B, as in Figure 17, the particle will take the least time sliding from A to B if the curve is part of an inverted arch of a cycloid.

The Dutch physicist Huygens had already shown by 1673 that the cycloid is also the solution to the **tautochrone problem**; that is, no matter where a particle P is placed on an inverted cycloid, it takes the same time to slide to the bottom (see Figure 18). Huygens proposed that pendulum clocks (which he invented) should swing in cycloidal arcs because then the pendulum would take the same time to make a complete oscillation whether it swings through a wide arc or a small arc.

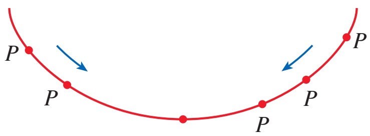


FIGURE 18

■ Families of Parametric Curves

EXAMPLE 10 Investigate the family of curves with parametric equations

$$x = a + \cos t \qquad y = a\tan t + \sin t$$

What do these curves have in common? How does the shape change as a increases?

SOLUTION We use a graphing calculator (or computer) to produce the graphs for the cases $a = -2, -1, -0.5, -0.2, 0, 0.5, 1$, and 2 shown in Figure 19. Notice that all of these curves (except the case $a = 0$) have two branches, and both branches approach the vertical asymptote $x = a$ as x approaches a from the left or right.